MTH

GS-2017 (Mathematics)

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Written Test in **MATHEMATICS - December 11, 2016**

For the Ph.D. Programs at TIFR, Mumbai and CAM & ICTS, Bangalore and for the Int. Ph.D. Programs at TIFR, Mumbai and CAM, Bangalore.

Duration : Three hours (3 hours)

Name : ____ Ref. Code : ____________

Please read all instructions carefully before you attempt the questions.

1. Please fill in details about name, reference code etc. on the answer sheet for Part I as well on the answer booklet of Part II . The Answer Sheet for Part I is machine-readable. Use only Black/Blue ball point pen to fill in the answer sheet.

2. **PART I** - There are thirty (30) True/False type questions in Part I of the question paper. Allotted time for Part I is 90 minutes. The answer sheet for Part I will be collected at the end of 90 minutes. Part I questions carry +2 for a correct answer, -1 (negative marks) for a wrong answer and 0 for not answering.

   Indicate your answer ON THE ANSWER SHEET by blackening the appropriate circle for each question. Do not mark more than one circle for any question : this will be treated as a wrong answer.

   We advise you to first mark the correct answers on the QUESTION PAPER and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.

3. **PART II** – 10 problems to be solved. The solutions should be written in the Answer Booklet for Part II that is provided. Extra blank sheets will be provided if needed. All Part II questions carry equal marks, and there are no negative marks. Partial credit will be given for partial solutions.

   Candidate can begin answering questions on Part II anytime. The answer booklet for Part II will be collected at the end of the exam.

4. **Selection Procedure** : The answers for Part I will be machine-graded. Part I will score will be used to decide a cut-off. Answer papers for Part II will be graded only for those candidates whose score is above the cut-off. List of candidates to be called for interview for the final selection for admission in the various programs will be decided based on the combined performance in Part I and II, weighted appropriately for each program.

5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an invigilator.

6. **Use of calculators, mobile phones, laptops, tablets (or other electronic devices, including those connecting to the internet) is NOT permitted.**

7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilators will announce it publicly.

8. Notation and Conventions used in this test are given on page 2 of the question paper.

Mathematics Question Paper, GS2017
Parts I and II

Notation and Conventions:

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, 3, \cdots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rationals, $\mathbb{R}$ the set of real numbers and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are assumed to carry the induced topology and metric. For a vector $v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$, the norm $||v||$ is defined by $||v||^2 = v_1^2 + \cdots + v_n^2$.
- $M_n(\mathbb{R})$ denotes the real vector space of $n \times n$ real matrices with the Euclidean metric.
- All logarithms are natural logarithms.

Part I

Answer whether the following statements are True or False. Mark your answer on the machince checkable answer sheet that is provided.

Note: $+2$ marks for a correct answer, -1 mark (negative marks) for a wrong answer, 0 marks for not answering.

1. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(x) \geq x^3$ for all $x \in [0, 1]$ with $\int_0^1 f(x)dx = \frac{1}{4}$. Then $f(x) = x^3$ for all $x \in \mathbb{R}$. True

2. Suppose $a,\ b,\ c$ are positive real numbers such that True

$$(1 + a + b + c)\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 16.$$

Then $a + b + c = 3$.

3. There exists a function $f : \mathbb{R} \to \mathbb{R}$ satisfying, False

$$f(-1) = -1, \quad f(1) = 1 \quad \text{and} \quad |f(x) - f(y)| \leq |x - y|^{\frac{3}{2}}, \quad \text{for all} \quad x, y \in \mathbb{R}.$$

4. Over the real line, False

$$\lim_{x\to\infty} \log \left(1+\sqrt{4+x}-\sqrt{1+x}\right) = \log(2).$$

5. Suppose f is a continuously differentiable function on $\mathbb{R}$ such that $f(x) \to 1$ and $f'(x) \to b$ as $x \to \infty$. Then $b = 1$. False

6. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and bijective, then f^{-1} is also differentiable. False

7. Let $H_1,\ H_2,\ H_3,\ H_4$ be four hyperplanes in $\mathbb{R}^3$. The maximum possible number of connected components of $\mathbb{R}^3 - (H_1 \cup H_2 \cup H_3 \cup H_4)$ is 14. False

8. Let $n \geq 2$ be a natural number. Let S be the set of all $n \times n$ real matrices whose entries are only 0, 1 or 2. Then the average determinant of a matrix in S is greater than or equal to 1. False

9. For any metric space (X, d) with X finite, there exists an isometric embedding $f : X \to \mathbb{R}^4$. False

10. There exists a non-negative continuous function $f : [0, 1] \to \mathbb{R}$ such that $\int_0^1 f^n dx \to 2$ as $n \to \infty$. False

11. There exists a subset A of $\mathbb{N}$ with exactly five elements such that the sum of any three elements of A is a prime number. False

12. There exists a finite abelian group G containing exactly 60 elements of order 2. False

13. Let $\alpha,\ \beta$ be complex numbers with non-positive real parts. Then True

$$|e^{\alpha} - e^{\beta}| \leq |\alpha - \beta|.$$

14. Every 2×2-matrix over $\mathbb{C}$ is a square of some matrix. False

15. Under the projection map $\mathbb{R}^2 \to \mathbb{R}$ sending (x, y) to x, the image of any closed set is closed. False

16. The number of ways a 2×8 rectangle can be tiled with rectangular tiles of size 2×1 is 34. True

17. Over the real line, True

$$\lim_{x\to\infty} \left(\frac{x + \log 9}{x - \log 9}\right)^x = 81.$$

18. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function with $\lim_{x\to\infty} f(x) = 0$. Then f has a maximum value in $[0, \infty)$. False

19. Given a continuous function $f : \mathbb{Q} \to \mathbb{Q}$, there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that the restriction of g to $\mathbb{Q}$ is f. False

20. For all positive integers m and n, if A is an $m \times n$ real matrix, and B is an $n \times m$ real matrix such that $AB = I$, then $BA = I$. False

21. There is a continuous onto function $f : S^2 \to S^1$ from the unit sphere in $\mathbb{R}^3$ to the unit sphere in $\mathbb{R}^2$, where $S^n = \{v \in \mathbb{R}^{n+1} \mid ||v|| = 1\}$ denotes the unit sphere in $\mathbb{R}^{n+1}$. True

22. Let P be a monic, non-zero, polynomial of even degree, and $K > 0$. Then the function $P(x) - Ke^x$ has a real zero. True

23. A p-Sylow subgroup of the underlying additive group of a finite commutative ring R is an ideal in R. True

24. Suppose A is an $n \times n$-real matrix, all whose eigenvalues have absolute value less than 1. Then for any $v \in \mathbb{R}^n$, $||Av|| \leq ||v||$. False

25. For any $x \in \mathbb{R}$, the sequence $\{a_n\}$, where $a_1 = x$ and $a_{n+1} = \cos(a_n)$ for all n, is convergent. True

26. Suppose $A_1, \cdots, A_m$ are distinct $n \times n$ real matrices such that $A_i A_j = 0$ for all $i \neq j$. Then $m \leq n$. False

27. In the symmetric group S_n any two elements of the same order are conjugate. False

28. If a particle moving on the Euclidean line traverses distance 1 in time 1 starting and ending at rest, then at some time $t \in [0, 1]$, the absolute value of its acceleration should be at least 4. True

29. Let $y(t)$ be a real valued function defined on the real line such that $y' = y(1 - y)$, with $y(0) \in [0, 1]$. Then $\lim_{t\to\infty} y(t) = 1$. False

30. The matrices

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x & 1 \\ 0 & y \end{pmatrix}, \quad x \neq y,$$

True

for any $x,\ y \in \mathbb{R}$ are conjugate in $M_2(\mathbb{R})$.

Part II

Write your solutions in the answer booklet provided. All questions carry equal marks. There are no negative marks, and partial credit will be given for partial solutions.

1. Show that the subset $GL_n(\mathbb{R})$ of $M_n(\mathbb{R})$ consisting of all invertible matrices is dense in $M_n(\mathbb{R})$.

2. Let f be a continuous function on $\mathbb{R}$ satisfying the relation

$$f(f(f(x))) = x \quad \text{for all } x \in \mathbb{R}.$$

Prove or disprove that f is the identity function.

3. Prove or disprove: the group of positive rationals under multiplication is isomorphic to its subgroup consisting of rationals which can be expressed as p/q, where both p and q are odd positive integers.

4. Show that the only elements in $M_n(\mathbb{R})$ commuting with every idempotent matrix are the scalar matrices. (A matrix P in $M_n(\mathbb{R})$ is said to be idempotent if $P^2 = P$.)

5. Prove or disprove the following: let $f : X \to X$ be a continuous function from a complete metric space (X, d) into itself such that $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$. Then f has a fixed point.

6. How many isomorphism classes of associative rings (with identity) are there with 35 elements? Prove your answer.

7. Prove or disprove: If G is a finite group and $g, h \in G$, then g, h have the same order if and only if there exists a group H containing G such that g and h are conjugate in H.

8. Prove or disprove: there exists $A \subset \mathbb{N}$ with exactly five elements, such that sum of any three elements of A is a prime number.

9. Show that there does not exist any continuous function $f : \mathbb{R} \to \mathbb{R}$ that takes every value exactly twice.

10. For which positive integers n does there exist a $\mathbb{R}$-linear ring homomorphism $f : \mathbb{C} \to M_n(\mathbb{R})$? Justify your answer.